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OSCILLATIONS OF A RIGID BODY WITH A TOROIDAL CAVITY FILLED WITH A VISCOUS LIQUID*

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An investigation is carried out of the damping of planar oscillations of a rigid body in which there is a toroidal cavity filled with a liquid of arbitrary viscosity. The diameter of the torus is assumed to be significantly larger than the diameter of the tube from which it is formed. On the assumption that the moment of inertia of the liquid is much smaller than that of the rigid body, an analytical expression is derived for the amplitude of the oscillations of the rigid body, and the optimal damping parameters are determined. For the case of a liquid with high viscosity the solution is compared with the well-known asymptotic solution, describing the oscillations of a rigid body with a cavity of arbitrary shape filled with a viscous liquid at small Reynolds numbers.

Problems concerning the motion of a body with a viscous liquid contained in thin tubes were considered by Gromeka and Joukowski. In recent years these problems have aroused renewed interest in connection with the use of tube dampers filled with a viscous liquid to suppress oscillations in spacecraft /1-9/. The most commonly used cavity shape in such dampers is toroidal.

1. Statement of the problem. We consider the planar oscillations of a rigid body, in which there is a toroidal cavity entirely filled with a viscous incompressible liquid of density ρ_*, about an axis parallel to the axis of the torus. To simplify matters we will assume that the centre of the torus is at the centre of mass of the system or at a fixed (stationary) point, if the latter exists. We shall assume that $\varepsilon = a/R \ll 1$, where R is the radius of the torus and a the radius of the tube forming the torus.

We introduce a cylindrical system of coordinates with its origin at the centre of the torus, the z axis directed along the axis of the torus and the coordinate lines r and φ in a plane perpendicular to the z axis. For small ε the components of the absolute velocity vector V satisfy the conditions $V_r \ll V_\varphi$, $V_z \ll V_\varphi$. We will therefore drop the terms containing V_r, V_z in the Navier-Stokes equations for V_φ. Then the equations of motion for the body with the liquid become

$$A\frac{d^2\Phi}{dt^2} + M\sin\Phi = N(V_\varphi) \tag{1.1}$$

$$\frac{\partial V_\varphi}{\partial t} = \nu\left(\frac{\partial^2 V_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial V_\varphi}{\partial r} + \frac{\partial^2 V_\varphi}{\partial z^2} - \frac{V_\varphi}{r^2}\right), \quad V_\varphi|_S = \frac{d\Phi}{dt} r|_S$$

where A is the principal central moment of inertia of the rigid body about the z axis, Φ is the angle of rotation of the body about the z axis, $M\sin\Phi$ is the restoring torque, N is the moment of the forces exerted by the liquid on the body, ν is the kinematic viscosity of the liquid, and S is the surface if the torus.

System (1.1) will subsequently be reduced to an integrodifferential equation describing the oscillations of the rigid body.

2. Integrodifferential equation. We will transform the coordinates by putting $r = x + R$ in the last two equations of (1.1) and reduce the system to non-dimensional form by means of the substitutions

$$\tau = \omega t, \quad u = V_\varphi/(\omega R), \quad \xi = x/a, \quad \zeta = z/a, \quad \omega^2 = M/A$$

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*Prikl. Matem. Mekhan., 54, 2, 201-206, 1990 where ω is the frequency of small oscillations of the rigid body. We obtain

$$\Phi^{\cdot\cdot} + \sin\Phi = N/(A\omega^2) \tag{2.1}$$

$$u^{\cdot} = \frac{\nu}{\omega a^2}\left[\frac{\partial^2 u}{\partial \xi^2} + \left(\frac{\varepsilon}{1+\varepsilon\xi}\right)\frac{\partial u}{\partial \xi} + \frac{\partial^2 u}{\partial \zeta^2} - \left(\frac{\varepsilon}{1+\varepsilon\xi}\right)^2 u\right]$$
$$u|_{\xi^2+\zeta^2=1} = \Phi^{\cdot}(1+\varepsilon\xi) \tag{2.2}$$

where, as before, $\varepsilon = a/R$; the dot denotes differentiation with respect to τ.

If the solution of the boundary-value problem (2.2) is sought as a series in powers of ε, the expression for the principal terms is

$$u^{\cdot} = \nu(\omega a^2)^{-1}(\partial^2 u/\partial\xi^2 + \partial^2 u/\partial\zeta^2), \quad u|_{\xi^2+\zeta^2=1} = \Phi^{\cdot} \tag{2.3}$$

We shall seek a solution of problem (2.3) satisfying the initial condition $u|_{t=0} = 0$. In the ξ, ζ plane we introduce coordinate ρ, ϑ by putting $\xi = \rho\cos\vartheta$, $\zeta = \rho\sin\vartheta$. Thanks to the symmetry of problem (2.3), for zero initial data u depends only on ρ and τ, and Eqs. (2.3) become

$$u^{\cdot} = \nu_0(\partial^2 u/\partial\rho^2 + \rho^{-1}\partial u/\partial\rho), \quad u|_{\rho=1} = u_S; \quad \nu_0 = \nu/(\omega a^2), \quad u_S = \Phi^{\cdot} \tag{2.4}$$

Following Joukowski's approach, we will find the solution of problem (2.4) such that $u_S = 1$, and then use the Duhamel integral /10, 11/.

We now take a Laplace transformation of (2.4). The solution of Eq.(2.4) in transforms, which is bounded at $\rho = 0$ and satisfies the boundary condition, has the form

$$u_*(\rho, p) = I_0(\rho\sqrt{p/\nu_0})/I_0(\sqrt{p/\nu_0})$$

where I_0 is the Bessel function of an imaginary argument of order zero and p is the transformation parameter.

By the inversion formula, we have

$$u(\rho,\tau) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{p}\exp(p\tau)\, u_*(\rho, p)\, dp \tag{2.5}$$

The singular points of the integrand are a pole at zero and poles at a denumerable sequence of points p_k - the roots of the equation $I_0(\sqrt{p_k/\nu_0}) = 0$. We will use the following notation: c_0 is the residue of the integrand at $p = 0$ and c_k is the residue at p_k. Noting that $I_0(x) \to 1$ as $x \to 0$ and changing to Bessel functions of a real argument of zero and first orders, we obtain

$$c_0 = 1, \quad c_k = -2\exp(-\lambda_k^2\nu_0\tau)\, J_0(\rho\lambda_k)/[\lambda_k J_1(\lambda_k)] \tag{2.6}$$

where λ_k are the zeros of J_0. Eqs.(2.5) and (2.6) yield the solution of the boundary-value problem (2.4) when $u_S = 1$. When $u_S = \Phi^{\cdot}$ we use the Duhamel integral to obtain

$$u(\rho,\tau) = \int_0^\tau \frac{d^2\Phi(\eta)}{d\eta^2}\left\{1 - 2\sum_{k=1}^\infty \exp[-\lambda_k^2\nu_0(\tau-\eta)]\frac{J_0(\rho\lambda_k)}{\lambda_k J_1(\lambda_k)}\right\} d\eta \tag{2.7}$$

In this approximation the viscous force exerted by the liquid on the torus wall per unit area is $\rho_*\nu\,[\partial V_\varphi/\partial(a\rho)]|_{\rho=1}$ (we recall that ρ_* is the density of the liquid) and the torque of the forces exerted by the liquid on the rigid body is $N = -4\pi^2R^3\omega\rho_*\nu\,(\partial u/\partial\rho)|_{\rho=1}$. Using the fact that $J_0' = -J_1$, we deduce from (2.7), (2.1) that

$$\Phi^{\cdot\cdot} + \sin\Phi = -\alpha\int_0^\tau \frac{d^2\Phi(\eta)}{d\eta^2}\Sigma(\tau-\eta)\, d\eta \tag{2.8}$$

$$\Sigma(\tau-\eta) = \sum_{k=1}^\infty \exp[-\lambda_k^2\nu_0(\tau-\eta)], \quad \alpha = \frac{8\pi^2R^3\rho_*\nu}{A\omega}, \quad \nu_0 = \frac{\nu}{\omega a^2}$$

3. *The case of a low mass of liquid.* To obtain a bound for the value g of the right-hand side of Eq.(2.8) we proceed as follows:

$$|g| \leqslant \alpha K\int_0^\tau \Sigma(\tau-\eta)\, d\eta = \alpha K\sum_{k=1}^\infty \frac{1}{\lambda_k^2\nu_0}[1 - \exp(-\lambda_k^2\nu_0\tau)] \leqslant$$

$$\frac{\alpha K}{\nu_0}\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}=\mu K;\quad K=\max|\Phi^{\cdot\cdot}|,\quad \mu=\frac{B}{A}$$

Here we have taken into account that /12/

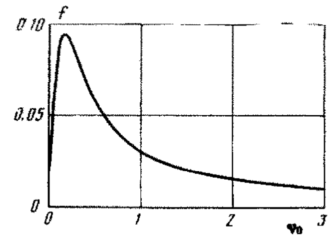
$$\lambda_1^{-2}+\lambda_2^{-2}+\ldots={}^1/_4,\quad \alpha/(4\nu_0)=B/A,\quad B=2\pi^2R^3a^2\rho_* \tag{3.1}$$

where B is the moment of inertia of the liquid about the axis of the torus.

On the other hand, it follows from (2.8) that $K\leqslant 1+|g|\leqslant 1+\mu K$, which yields $K\leqslant (1-\mu)^{-1}$. Thus $|g|\leqslant \mu(1-\mu)^{-1}$.

Let us assume that the mass of the liquid is significantly less than that of the body; then $\mu\ll 1$ and $g=O(\mu)$. We consider small oscillations. To analyse Eq.(2.8) we use the method of averaging.

In the unperturbed problem ($\mu=0$, $|\Phi|\ll 1$) we have



$$\Phi=a_*\sin\tau+b_*\cos\tau,$$
$$\Phi^{\cdot}=a_*\cos\tau-b_*\sin\tau$$

Choose a_*, b_* as new variables in problem (2.8). After some standard algebra, omitting terms of second order in μ on the right, we obtain

$$\frac{a_*^{\cdot}}{\alpha\cos\tau}=-\frac{b_*^{\cdot}}{\alpha\sin\tau}=\int_0^\tau[a_*(\eta)\sin\eta+b_*(\eta)\cos\eta]\,\Sigma(\tau-\eta)\,d\eta \tag{3.2}$$

Carrying out the integration and dropping terms of the second order in μ, we get

$$a_*^{\cdot}=\alpha\cos\tau\sum_{k=1}^{\infty}G_k(\tau),\quad b_*^{\cdot}=-\alpha\sin\tau\sum_{k=1}^{\infty}G_k(\tau) \tag{3.3}$$

$$G_k=\{b_*(\tau)\sin\tau-a_*(\tau)\cos\tau+\lambda_k^2\nu_0[a_*(\tau)\sin\tau+b_*(\tau)\cos\tau]+$$
$$[a_*(0)-\lambda_k^2\nu_0 b_*(0)]\exp(-\lambda_k^2\nu_0\tau)\}(1+\lambda_k^4\nu_0^2)^{-1}$$

Averaging the right-hand sides of Eqs.(3.3) with respect to τ from 0 to ∞, we find that

$$a_*^{\cdot}={}^1/_2\alpha(-a_*\Sigma_1+b_*\Sigma_2),\quad b_*^{\cdot}={}^1/_2\alpha(-a_*\Sigma_2-b_*\Sigma_1) \tag{3.4}$$

$$\Sigma_1=\sum_{k=1}^{\infty}\frac{1}{1+\lambda_k^4\nu_0^2},\quad \Sigma_2=\sum_{k=1}^{\infty}\frac{\lambda_k^2\nu_0}{1+\lambda_k^4\nu_0^2}$$

Integrating these equations with due allowance for the value of α as in (2.8), we obtain the following expression for the amplitude of the oscillations:

$$F=(a_*^2+b_*^2)^{1/2}=F_0\exp\left(-2\frac{B}{A}f\omega t\right),\quad f=\sum_{k=1}^{\infty}\frac{\nu_0}{1+\lambda_k^4\nu_0^2},\ \nu_0=\frac{\nu}{\omega a^2} \tag{3.5}$$

The damper will be optimal if f is a maximum. The figure represents the function $f(\nu_0)$. The optimum value of the non-dimensional viscosity ν_0 is 0.158.

4. The case of a strongly viscous liquid. Let us assume now that ($\nu_0\gg 1$). From (3.5) we derive the equality

$$f=\frac{1}{32\nu_0}+O\left(\frac{1}{\nu_0^3}\right),\quad F=F_0\exp\left(-\frac{\omega^2a^2B}{16\nu A}t\right) \tag{4.1}$$

(using the fact that /12/ $\lambda_1^{-4}+\lambda_2^{-4}+\ldots=1/32$).

On the other hand, it was shown in /13/ that in the case of a strongly viscous liquid small oscillations of our system are described by the equation

$$J\partial^2\Phi/\partial t^2+M\Phi=-\rho_*P(\nu J)^{-1}M\partial\Phi/\partial t \tag{4.2}$$

where $J=A+B$ is the moment of inertia of the body with the liquid, and P is a component of a certain tensor which depends only on the shape of the cavity. For a torus this component is

$$P = -2\pi \int_D r^2 W(r, z)\, dr dz, \quad D: \{(r-R)^2 + z^2 \leqslant a^2\} \tag{4.3}$$

Here W is the solution of the boundary-value problem (Γ is the boundary of the disk D)

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} - \frac{W}{r^2} + \frac{\partial^2 W}{\partial z^2} = r, \quad W|_\Gamma = 0 \tag{4.4}$$

We shall now work out an approximate expression for P in the case of a torus with $\varepsilon = a/R \ll 1$ and compare the solution of Eq.(4.2) with (4.1).

In (4.4) we make the substitution

$$r = x + R \tag{4.5}$$

and transform to non-dimensional variables via the transformation

$$W = a^3\Omega, \quad x = a\xi, \quad z = a\zeta \tag{4.6}$$

This gives

$$\Delta\Omega + \left(\frac{\varepsilon}{1+\varepsilon\xi}\right)\frac{\partial\Omega}{\partial\xi} - \left(\frac{\varepsilon}{1+\varepsilon\xi}\right)^2\Omega = \frac{1}{\varepsilon} + \xi \tag{4.7}$$
$$\Omega|_\Gamma = 0, \ \Gamma: \{\xi^2 + \zeta^2 = 1\}$$

We will seek a solution of the boundary-value problem (4.7) as a series in powers of the small parameter ε:

$$\Omega(\xi, \zeta) = \varepsilon^{-1}\Omega_{-1}(\xi, \zeta) + \Omega_0(\xi, \zeta) + \varepsilon\Omega_1(\xi, \zeta) + \varepsilon^2\Omega_2(\xi, \zeta) + \ldots \tag{4.8}$$

$$\Omega_k|_\Gamma = 0, \quad k = -1, 0, 1, 2, \ldots \tag{4.9}$$

Substituting (4.8) into Eq.(4.7), expanding the coefficients of the latter in powers of ε and equating coefficients of like powers, we obtain

$$\Delta\Omega_{-1} = 1 \tag{4.10}$$
$$\Delta\Omega_0 = -\partial\Omega_{-1}/\partial\xi + \xi \tag{4.11}$$
$$\Delta\Omega_1 = \xi(\partial\Omega_{-1}/\partial\xi) - \partial\Omega_0/\partial\xi + \Omega_{-1} \tag{4.12}$$
$$\Delta\Omega_2 = -\xi^2(\partial\Omega_{-1}/\partial\xi) + \xi(\partial\Omega_0/\partial\xi) - \partial\Omega_1/\partial\xi - 2\xi\Omega_{-1} + \Omega_0 \tag{4.13}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

with the boundary condition (4.9).

The boundary-value problem (4.10), (4.9) describes Poiseuille flow. Substituting its solution $\Omega_{-1} = (\rho^2 - 1)/4$, $\rho^2 = \xi^2 + \zeta^2$, into (4.11) we obtain

$$\Delta\Omega_0 = \xi/2 \tag{4.14}$$

Solutions of this and the following boundary-value problems satisfying coditions (4.9) can be sought as linear combinations of the functions

$$(\rho^2 - 1)^n\xi^m, \quad n = 1, 2, \ldots, \quad m = 0, 1, 2, \ldots$$

Omitting the details, we will present the final results. The solution of problem (4.14), (4.9) is

$$\Omega_0 = (\rho^2 - 1)\,\xi/16 \tag{4.15}$$

Substituting (4.15) into (4.12) and solving the corresponding boundary-value problem, we get

$$\Omega_1 = -(\rho^2 - 1)/64 + (\rho^2 - 1)^2/128 + (\rho^2 - 1)\,\xi^2/32 \tag{4.16}$$

Finally, substituting (4.16) into (4.13), we find that

$$\Omega_2 = \frac{17}{1024}(\rho^2 - 1)\xi - \frac{7}{256}(\rho^2 - 1)\xi^3 - \frac{13}{1024}(\rho^2 - 1)^2\xi$$

We make the substitutions (4.5), (4.6) in the integral (4.3) and transform to polar coordinates $\xi = \rho\cos\vartheta$, $\zeta = \rho\sin\vartheta$. Using the series (4.8), we obtain

$$P = -2\pi a^5 R^2 \int_0^1 \int_0^{2\pi} (1 + \varepsilon\rho\cos\vartheta)^2\, \Omega\rho\, d\rho\, d\vartheta = \pi^2 a^5 R^2 \left[\frac{1}{4\varepsilon} + \frac{3\varepsilon}{64} + O(\varepsilon^3)\right] \tag{4.17}$$

It should be mentioned that a representation of the tensor component characterizing the effect of a strongly viscous liquid on the motion of the body /13/ has been derived for a torus with arbitrary ratio $\varepsilon = a/R$ /2/. This representation takes the form of an infinite series of improper integrals of special functions, with coefficients that are roots of certain transcendental equations. Another available result /2/ is the first term in the asymptotic expansion of the tensor for small ε - for P this is identical with the first term in (4.17).

Eq.(4.2) describes damped oscillations with amplitude

$$F = F_0 \exp\left(-\frac{\rho_* MP}{2\nu J^2} t\right) \tag{4.18}$$

Substituting the principal term of the expansion (4.17) into (4.18), with allowance for the expression (3.1) for the moment of inertia B of the liquid and the fact that the quotient B/A is small, we finally arrive at (4.1).

5. *Example.* Consider the damping of the oscillations of a small space satellite stabilized by the Earth's magnetic field. Let $A = 5\,\mathrm{kg \cdot m^2}$ and let the characteristic frequency of oscillations about the force line be $\omega = 0.026\ \mathrm{sec}^{-1}$ (a period equal to 4 min). Assume that the radius of the toroidal tube a and the kinematic viscosity ν are chosen optimally, i.e., in (3.5) $\nu_0 = \nu/(\omega a^2) = 0.158$. Then the effect of the damper will be maximal when the actual moment of inertia B of the liquid is a maximum. The value of B is bounded by the admissible mass of liquid and the dimensions of the torus, which depend on the conditions in the spacecraft. In order to cut down the volume of the damper, a liquid of maximum possible density should be used.

Let the admissible radius of the torus be $R = 0.1$ m, and suppose that the selected damping liquid is mercury: $\nu = 0.11 \times 10^{-6}\ \mathrm{m^2/sec}$. Then the optimum value of ν_0 is obtained at a tube diameter $2a = 1$ cm. The moment of inertia B of the liquid is 0.0067 $\mathrm{kg \cdot m^2}$ (with a mass of liquid of 0.67 kg). Substituting these quantities into (3.5) we obtain a damping constant of 1.5×10^5 sec.

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